Acta Sci. Math., 45 (1983), 279—280

Dilation theory and one-parameter semigroups of contractions

ISTVÁN KOVÁCS

To Professor Béla Szőkefalvi-Nagy on his 70th birthday

In the frame of the dilation theory, a classification of contraction operators, briefly contractions, T on a Hilbert space $\mathfrak{H}$, according to the behavior of their powers T^n and T^{*n} as $n \to \infty$, was given in [2], Chap. II in terms of the classes $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$ $(\alpha, \beta = 0, 1)$. Analogously, we may consider classes of one-parameter semigroups of contractions (T_t), i.e., representations $t \mapsto T_t$ of the additive semigroup $\mathbf{R}_+$ of the non-negative reals by contractions T_t of $\mathfrak{H}$, defined as follows:

$$SC_{0\cdot} = \{(T_t)\colon\ T_t x \to 0 \quad \text{for all } x \in \mathfrak{H}\},$$
$$SC_{\cdot 0} = \{(T_t)\colon\ T_t^* x \to 0 \quad \text{for all } x \in \mathfrak{H}\},$$
$$SC_{1\cdot} = \{(T_t)\colon\ T_t x \to 0 \quad \text{for } \ x = 0\ (x \in \mathfrak{H}) \text{ only}\},$$
$$SC_{\cdot 1} = \{(T_t)\colon\ T_t^* x \to 0 \quad \text{for } \ x = 0\ (x \in \mathfrak{H}) \text{ only}\},$$

whenever $t \to \infty$. Furthermore, set $SC_{\alpha\beta} = SC_{\alpha\cdot} \cap SC_{\cdot\beta}$ $(\alpha, \beta = 0, 1)$.

It might be an interesting question to know to what extent results and facts derived for the elements of $C_{\alpha\beta}$ would hold true for the elements of $SC_{\alpha\beta}$, for instance in the sense that is shown by the following observation.

One of the consequences of the paper [1] is that a contraction T which is an element of a finite-type von Neumann algebra $\mathscr{A}$ (cf. [3], Chap. V) is a unitary operator if and only if it belongs to the class $C_{1\cdot}$. According to the above considerations, a result analogous to that one might sound like this.

Theorem. *A one-parameter semigroup of contractions (T_t) the elements of which belong to a given finite-type von Neumann algebra $\mathscr{A}$ can be extended to a one-parameter group of unitary operators (U_t) of $\mathscr{A}$ if and only if $(T_t) \in SC_{1\cdot}$.*

Proof. The necessity part of the proof is evident. To show that the condition is also sufficient, consider the family of operators $F = (T_t^* T_t)_{t \in \mathbf{R}_+}$. This family

Received May 18, 1982.

is evidently bounded from below and is also decreasing as $t\to\infty$. In fact, if $s<t$ then as $(t-s)>0$ and $t=(t-s)+s$, for every $x\in\mathfrak{H}$ *) we have

$$0 \leqq (T_t^* T_t x|x) = (T_t x|T_t x) = (T_{t-s}T_s x|T_{t-s}T_s x) \leqq \|T_{t-s}\|^2 (T_s x|T_s x) \leqq (T_s^* T_s x|x).$$

So F has a strong cluster point S in $\mathscr{A}$ which is self-adjoint and positive:

$$S = \lim\nolimits_{\text{strong}} T_t^* T_t \quad \text{as} \quad t \to \infty.$$

Moreover, S is invertible in the more general sense: $x\in\mathfrak{H}$, $Sx=0$ imply $x=0$. This follows from the fact that

$$\lim_{t\to\infty} \|T_t x\|^2 = \lim_{t\to\infty} (T_t^* T_t x|x) = (Sx|x) = 0,$$

which gives that $x=0$ by the condition of the theorem. Furthermore, for every $s\in\mathbf{R}_+$ we have $T_s^* S T_s = S$. This is immediate by the definition of S. Then for every finite normal trace φ on $\mathscr{A}$ we have $\varphi(S)=\varphi(T_s^* S T_s)=\varphi(S^{1/2}T_s T_s^* S^{1/2})$, from which we may conclude

$$(*) \qquad \varphi(S^{1/2}(I-T_s T_s^*)S^{1/2}) = 0$$

(I is the identity operator of $\mathfrak{H}$). As $S^{1/2}(I-T_s T_s^*)S^{1/2} \geqq O$ and φ is arbitrary, $(*)$ implies $S^{1/2}(I-T_s T_s^*)S^{1/2}=O$. Now, with S its square root $S^{1/2}$ is also invertible and its inverse is densely defined. Therefore, $S^{1/2}(I-T_s T_s^*)S^{1/2}=O$ implies $I-T_s T_s^*=O$, i.e., T_s^* is an isometric operator. But $\mathscr{A}$ is of finite-type, thus T_s^* and hence T_s are both unitary operators. To complete the proof, set

$$U_t = \begin{cases} T_t & \text{if} \quad t \geqq 0, \\ T_t^* & \text{if} \quad t < 0. \end{cases}$$

References

[1] C. Foiaş—I. Kovács, Une caractérisation nouvelle des algèbres de von Neumann finies, *Acta Sci. Math.*, **23** (1962), 274—278.

[2] B. Sz.-Nagy—C. Foiaş, *Analyse harmonique des opérateurs dans l'espace de Hilbert*, Akadémiai Kiadó (Budapest, 1967) et Masson (Paris, 1967).

[3] M. Takesaki, *Theory of Operator Algebras. I*, Springer-Verlag (Berlin—New York—Heidelberg, 1979).

DEPARTMENT OF MATHEMATICS & STATISTICS
UNIVERSITY OF SOUTH ALABAMA
MOBILE, AL 36688, U.S.A.

*) $\mathfrak{H}$ is the underlying Hilbert space of $\mathscr{A}$ and the operators under consideration.